THEORY OF MULTIPLICATIVE PARTITIONS

Factorisatio Numerorum

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Introduction. Here I explore an interrelated set of mathematical questions that arise whenever one looks (as I recently have been doing) to the quantum theory of composite systems, at least when the quantum states can be assumed to live in finite-dimensional Hilbert spaces. To pose the questions I have in mind I look to some simple examples.

Let

$$\mathsf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \mathsf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \mathsf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Then

$$\mathsf{a} \otimes \mathsf{b} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_1 b_3 \\ a_2 b_1 \\ a_2 b_2 \\ a_2 b_3 \end{pmatrix}, \quad \mathsf{b} \otimes \mathsf{a} = \begin{pmatrix} b_1 a_1 \\ b_1 a_2 \\ b_2 a_1 \\ b_2 a_2 \\ b_3 a_1 \\ b_3 a_2 \end{pmatrix}$$

We note in passing that if a and b are unit vectors $\mathsf{a}^+\mathsf{a} = \mathsf{b}^+\mathsf{b} = 1$ then so also are their tensor products: $(\mathsf{a} \otimes \mathsf{b})^+(\mathsf{a} \otimes \mathsf{b}) = (\mathsf{a}^+\mathsf{a}) \otimes (\mathsf{b}^+\mathsf{b}) = 1 \otimes 1 = 1$.[1]

Our root problem is a *decision problem*, which in the simplest instance can be posed: Given a 6-vector

$$\boldsymbol{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{pmatrix}$$

do there exist vectors a and b such that

$$\boldsymbol{\Phi} = \mathsf{a} \otimes \mathsf{b} \quad \text{else} \quad \mathsf{b} \otimes \mathsf{a}$$

In short, is $\boldsymbol{\Phi}$ "separable," or is it "non-separable"? Suppose $\boldsymbol{\Phi}$ (normalized) refers to the state of a composite system $\mathcal{S}_{AB} = \mathcal{S}_A \otimes \mathcal{S}_B$. In that context the question becomes: Are the states of $\mathcal{S}_A$ and $\mathcal{S}_B$ "disentangled" or "entangled"?

[1] I have made use here of properties of the tensor (Kronecker) product which are listed on page 24 of Chapter 1 in *Advanced Quantum Topics* (2000).

In the latter case, what does one *mean* by "the state of $\mathcal{S}_A$ (or $\mathcal{S}_B$)"? Here the *density matrix* and *partial trace* concepts come indispensably into play, and Dirac notation becomes the natural language. The pure state density operators

$$\begin{aligned}\rho_A &= |a)(a| \\ \rho_B &= |b)(b|\end{aligned}$$

are clearly projective ($\rho_A\rho_A = \rho_A,\ \rho_B\rho_B = \rho_B$) and have unit trace[2]

$$\mathrm{tr}\,\rho_A = \sum_i (e_i|a)(a|e_i) = \sum_i (a|e_i)(e_i|a) = (a|a) = 1$$

and the same can be said of $\rho_\varPhi = |\varPhi)(\varPhi|$. In separable cases we have

$$\begin{aligned}\rho_\varPhi = |\varPhi)(\varPhi| &= \big[|a)\otimes|b)\big]\big[(a|\otimes(b|\big] \\ &= \big[|a)(a|\big]\otimes\big[|b)(b|\big] \\ &= \rho_A\otimes\rho_B &&(1.1)\\ \mathrm{tr}\,\rho_\varPhi &= \mathrm{tr}\,\rho_A\cdot\mathrm{tr}\,\rho_B &&(1.2)\end{aligned}$$

But what can be said—more specifically, what becomes of (1)—when $\varPhi$ is *not* separable? It is here that the partial trace comes into play.

In $6 = 2\cdot 3 = 3\cdot 2$ dimensions, four distinct partial trace operations are available:[3]

$$\begin{aligned}
\mathrm{tr}_{2(3)}\mathbb{X} &= \sum_{i=1}^{2}\big[(e_i|\otimes\mathbb{I}_3\big]\mathbb{X}\big[|e_i)\otimes\mathbb{I}_3\big] &&:\quad \text{trace out leading } 2\times 2 \text{ component}\\
\mathrm{tr}_{(2)3}\mathbb{X} &= \sum_{j=1}^{3}\big[\mathbb{I}_2\otimes(f_j|\big]\mathbb{X}\big[\mathbb{I}_2\otimes|f_j)\big] &&:\quad \text{trace out trailing } 3\times 3 \text{ component}\\
\mathrm{tr}_{(3)2}\mathbb{X} &= \sum_{i=1}^{2}\big[\mathbb{I}_3\otimes(e_i|\big]\mathbb{X}\big[\mathbb{I}_3\otimes|e_i)\big] &&:\quad \text{trace out trailing } 2\times 2 \text{ component}\\
\mathrm{tr}_{3(2)}\mathbb{X} &= \sum_{j=1}^{3}\big[(f_j|\otimes\mathbb{I}_2\big]\mathbb{X}\big[|f_j)\otimes\mathbb{I}_2\big] &&:\quad \text{trace out leading } 3\times 3 \text{ component}
\end{aligned}$$

The composite systems mosta commonly encountered in quantum theory (think of the adventures of Alice and Bob, or of systems in interaction with their environments) are bipartite, and factor order is arbitrarily preassigned. In such contexts is simpler and more natural to write

$$\begin{aligned}\mathrm{tr}_A\mathbb{X} &= \mathrm{tr}_{2(3)}\mathbb{X}\\ \mathrm{tr}_B\mathbb{X} &= \mathrm{tr}_{(2)3}\mathbb{X}\end{aligned}$$

and it is in that notation that I phrase most of the following remarks.

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[2] Here $\{|e_i)\}$ refers to some/any orthonormal basis in $\mathcal{H}_A$.

[3] Here $\{|e_i)\}$ refers to some/any orthonormal basis in $\mathcal{H}_2$ and $\{|f_j)\}$ refers to some/any orthonormal basis in $\mathcal{H}_3$.

Introducing matrices

$$\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbb{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

we have

$$\begin{aligned} \mathrm{tr}_A(\mathbb{A}\otimes\mathbb{B}) &= (\mathrm{tr}\,\mathbb{A})\,\mathbb{B} \\ \mathrm{tr}_B(\mathbb{A}\otimes\mathbb{B}) &= (\mathrm{tr}\,\mathbb{B})\,\mathbb{A} \end{aligned}$$

giving

$$\mathrm{tr}\,[\mathrm{tr}_A(\mathbb{A}\otimes\mathbb{B})] = \mathrm{tr}\,[\mathrm{tr}_B(\mathbb{A}\otimes\mathbb{B})] = (\mathrm{tr}\,\mathbb{A})(\mathrm{tr}\mathbb{B}) = \mathrm{tr}\,(\mathbb{A}\otimes\mathbb{B}) \tag{2}$$

and can recover $\mathbb{C} = \mathbb{A}\otimes\mathbb{B}$ from its trace and partial traces:

$$\mathbb{A}\otimes\mathbb{B} = \frac{\mathrm{tr}_B(\mathbb{A}\otimes\mathbb{B})\otimes\mathrm{tr}_A(\mathbb{A}\otimes\mathbb{B})}{\mathrm{tr}\,(\mathbb{A}\otimes\mathbb{B})} = \frac{\mathrm{tr}_B(\mathbb{A}\otimes\mathbb{B})}{\sqrt{\mathrm{tr}\,(\mathbb{A}\otimes\mathbb{B})}}\otimes\frac{\mathrm{tr}_A(\mathbb{A}\otimes\mathbb{B})}{\sqrt{\mathrm{tr}\,(\mathbb{A}\otimes\mathbb{B})}}$$

It becomes appropriate at this point to note that in two distinct respects the separability problem is non-unique:

- If $\mathbb{C}$ is separable then so is $k\mathbb{C}$, so (set $k = 1/\mathrm{tr}\,\mathbb{C}$) one can without loss of generality assume $\mathrm{tr}\,\mathbb{C} = 1$.

- $\mathbb{A}\otimes\mathbb{B} = (k\mathbb{A})\otimes(k^{-1}\mathbb{B})$, so if $\mathbb{C} = \mathbb{A}\otimes\mathbb{B}$ has unit trace one can without loss of generality assume that $\mathbb{A}$ and $\mathbb{B}$ also have unit trace:

$$\mathbb{A} = \frac{\mathrm{tr}_B(\mathbb{A}\otimes\mathbb{B})}{\mathrm{tr}\,[\mathrm{tr}_B(\mathbb{A}\otimes\mathbb{B})]} \quad \text{and} \quad \mathbb{B} = \frac{\mathrm{tr}_A(\mathbb{A}\otimes\mathbb{B})}{\mathrm{tr}\,[\mathrm{tr}_A(\mathbb{A}\otimes\mathbb{B})]}$$

Square matrices with unit traces[4] will be said to be "trace-normalized" (or simply "normalized" when the context precludes the possibility of confusion.)

For *any* 6×6 matric $\mathbb{C}$ one has[5] (compare (1))

$$\begin{aligned} \mathrm{tr}\,[\mathrm{tr}_A\mathbb{C}] = \mathrm{tr}\,[\mathrm{tr}_B\mathbb{C}] &= \mathrm{tr}\,\mathbb{C} \\ &= 1 \text{ if } \mathbb{C} \text{ is normalized} \end{aligned}$$

but one has[5]

$$\frac{\mathbb{C}}{\mathrm{tr}\,\mathbb{C}} = \frac{\mathrm{tr}_B\mathbb{C}}{\mathrm{tr}[\mathrm{tr}_B\mathbb{C}]}\otimes\frac{\mathrm{tr}_A\mathbb{C}}{\mathrm{tr}[\mathrm{tr}_A\mathbb{C}]} \quad \text{iff } \mathbb{C} \text{ is } \mathbb{A}\otimes\mathbb{B} \text{ separable} \tag{3}$$

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[4] But subject, however, to none of the other conditions (positive semi-definite hermiticity) imposed upon density matrices.

[5] Here as on some future occasions I allow myself to omit the formal demonstration if *Mathematica*-based numerical experimentation has convinced me of the validity of a claim.

The matrix $\mathbb{C} = \mathbb{B} \otimes \mathbb{A}$ is separable, but would fail the preceding test, though it would pass an obvious variant of that test. Similarly, one would need a suite of distinct partial trace tests to discover whether a 24×24 matrix $\mathbb{Z}$ is separable in one or another of the following senses

$$\begin{array}{ccc} \mathbb{A} \otimes \mathbb{B} \otimes \mathbb{C} & , & \mathbb{C} \otimes \mathbb{B} \otimes \mathbb{A} \\ \mathbb{B} \otimes \mathbb{C} \otimes \mathbb{A} & , & \mathbb{A} \otimes \mathbb{C} \otimes \mathbb{B} \\ \mathbb{C} \otimes \mathbb{A} \otimes \mathbb{B} & , & \mathbb{B} \otimes \mathbb{A} \otimes \mathbb{C} \end{array}$$

(here $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ have been assumed to be respectively 2, 3, and 4-dimensional). And still further testing would be required to discover whether $\mathbb{Z}$, having passed (say) the $\mathbb{A} \otimes \mathbb{B} \otimes \mathbb{C}$ test, possesses the still more highly composite structure

$$\mathbb{A} \otimes \mathbb{B} \otimes \mathbb{A} \otimes \mathbb{A}$$

Preceding remarks exemplify a problem which we discuss now in more general terms.

How many, and which? An n-vector (similarly, an $n \times n$ matrix) can be composite or "separable" only if n is composite in the number theoretic sense (not prime):

$$n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_\nu^{\alpha_\nu} \quad : \quad \textstyle\sum_i \alpha_i \neq 1 \tag{4}$$

We are interested in ennumerating the m-term factorizations of n. When I mentioned my interest in this problem to Scott Corry, a former student who is now a number theorist on the mathematics faculty at Lawrence University with whom I was in correspondence about other matters, he promptly responded[6] that I had ventured into an area known as "factorisatio numerorum," and cited several references.[7] Quoting from the introduction to another paper[8]

> *The problems of "factorisatio numerorum," which go back more than 65 years, are concerned principally with* (i) *the total number of factorizations of a natural number* $n > 1$ *into products of natural numbers larger than* 1*, where the order of the factors is not counted, and* (ii) *the corresponding total number* $F(n)$ *when the order of the factors is counted.*

Effort in the field appears to have been directed mainly to developing asymptotic upper and lower bounds on $f(n)$ and $F(n)$. My own need, however , is more particular.

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[6] Private communication, 20 February 2013.

[7] Among them E. R. Canfield, Paul Erdös & Carl Pomerance, "On a problem of Oppenheim concerning 'factorisatio numerorum'," Journal of Number Theory **17**, 1-28 (1983), which seems to be a classic in the field.

[8] Arnold Knoppmacher, John Knoppmacher & R. Warlimot, "'Factorisatio numerorum' in arithmetical semigroups," Acta Mathematica **61**, 327-336 (1992).

Factorizations of the form

$$n = q_1 q_2 \cdots q_m \tag{5}$$

will be said to be "m-partite." We might write

$$\begin{aligned} f(n,m) &= \text{total \# of } m\text{-partite factorizations of } n \text{ (order not counted)} \\ F(n,m) &= \text{total \# of } m\text{-partite factorizations of } n \text{ (order counted)} \end{aligned}$$

but are in fact not so much interested in "how many" as "which." We want to be in position to list (at least for n not too large) the *individual members of the population* of m-partite factorizations of n, both when order is not counted and (especially) when order is counted.

Some of the factors in (5) may appear multiple times. Lumping repeated factors together, we obtain

$$n = Q_1^{\beta_1} Q_2^{\beta_2} \cdots Q_k^{\beta_k} \quad : \quad \textstyle\sum_j \beta_j = m, \text{ all } Q_j \text{ distinct}$$

Suppose, for example, that as an instance of (5) we had

$$\begin{aligned} n &= abracadabra \\ &= a^5 b^2 c^1 d^1 r^2 \quad \text{if order is not counted} \end{aligned}$$

That 11-partite product (of which $2^5 \cdot 3^2 \cdot 4^1 \cdot 5^1 \cdot 6^2 = 207360 = 2^9 \cdot 3^4 \cdot 5$ provides an instance) admits of

$$\frac{11!}{5!2!1!1!2!} \equiv \binom{11}{5,2,1,1,2} = 83160$$

distinct permutations, of which one is *abracadabra*. Evidently, if we possessed a list of the m-partite factorizations of n (order not counted) we could by permutation generate the corresponding list in which order does count. It is clear that we can in typical cases expect to have expect to have

$$F(n,m) \gg f(n,m)$$

and why the production of asymptotic estimates is a challenging exercise.

A computational strategy. If, given $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_\nu^{\alpha_\nu}$, our objective is to construct an m-partite factorization $n = q_1 q_2 \cdots q_m$ we must partition the exponents and distribute their parts among the q-factors, in such a way that no factor is left empty-handed (all exponents 0), for such a factor would have value 1 and the resulting factorization would not be m-partite. Writing

$$q_k = p_1^{\mu_{k1}} p_2^{\mu_{k2}} p_3^{\mu_{k3}} \cdots p_\nu^{\mu_{k\nu}} \quad : \quad k = 1, 2, \ldots, m$$

our assignment is to ennumerate the solutions of

$$\left.\begin{aligned} \mu_{11}+\mu_{21}+\cdots+\mu_{m1}&=\alpha_1\\ \mu_{12}+\mu_{22}+\cdots+\mu_{m2}&=\alpha_2\\ &\vdots\\ \mu_{1\nu}+\mu_{2\nu}+\cdots+\mu_{m\nu}&=\alpha_\nu \end{aligned}\right\}\tag{6}$$

where the μs are drawn from $\mathbb{N}_0=\{0,1,2,\ldots\}$. Evidently, we can with every distinct m-partite factorization of $n=q_1q_2\cdots q_m=p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_\nu^{\alpha_\nu}$ associate a $\nu\times m$ matrix

$$\mathbb{M}=\begin{pmatrix}\mu_{11}&\mu_{21}&\cdots&\mu_{m1}\\ \mu_{12}&\mu_{22}&\cdots&\mu_{m2}\\ \vdots&\vdots&&\vdots\\ \mu_{1\nu}&\mu_{2\nu}&\cdots&\mu_{m\nu}\end{pmatrix}\quad:\quad\text{no column null}$$

where the elements of the j^{th} row comprise a partition of α_j, while the elements of the k^{th} column are the prime exponents that enter into the construction of q_m.

Mathematica is nicely equipped to assist in the calculations. Suppose we had in mind an n of the form

$$n=p_1^5p_2^2$$

of which the smallest instance is $2^5\cdot3^2=288$ (but the present argument looks only to the exponents, and is insensitive to the prime values we assign to p_1 and p_2), and we interested in the tripartite factorizations $q_1q_2q_3$ of n. There are 7 partitions of 5

$$\texttt{PartitionsP[5]}=7$$

and they are

$$\texttt{IntegerPartitions[5]}=\Big\{\{5\},\{4,1\},\{3,2\},\{3,1,1\},\{2,2,1\},\\ \{2,1,1,1\},\{1,1,1,1,1\}\Big\}$$

We seek a tripartite factorization, so have no interest in partitions into more than three terms; those are obvious in this instance, but might have been produced by these commands:

$$\begin{aligned}&\texttt{IntegerPartitions[5,\{1\}]}=\{5\}\\ &\texttt{IntegerPartitions[5,\{2\}]}=\{4,1\},\{3,2\}\\ &\texttt{IntegerPartitions[5,\{3\}]}=\{3,1,1\},\{2,2,1\}\end{aligned}$$

So we have these candidates

$$\begin{gathered}\{5,0,0\}\\ \{4,1,0\}\\ \{3,2,0\}\\ \{3,1,1\}\\ \{2,2,1\}\end{gathered}$$

for insertion into the top row of the 2×3 matrix $\mathbb{M}$, and similarly find these candidates

$$\begin{gathered}\{2,0,0\}\\\{1,1,0\}\end{gathered}$$

for insertion into the second row. We are in position therefore to consruct ten $\mathbb{M}$ matrices

$$\left.\begin{matrix}\begin{pmatrix}5&0&0\\2&0&0\end{pmatrix},\begin{pmatrix}4&1&0\\2&0&0\end{pmatrix},\begin{pmatrix}3&2&0\\2&0&0\end{pmatrix},\begin{pmatrix}3&1&1\\2&0&0\end{pmatrix},\begin{pmatrix}2&2&1\\2&0&0\end{pmatrix}\\\begin{pmatrix}5&0&0\\1&1&0\end{pmatrix},\begin{pmatrix}4&1&0\\1&1&0\end{pmatrix},\begin{pmatrix}3&2&0\\1&1&0\end{pmatrix},\begin{pmatrix}3&1&1\\1&1&0\end{pmatrix},\begin{pmatrix}2&2&1\\1&1&0\end{pmatrix}\end{matrix}\right\}\tag{7}$$

of which six have to be discarded because they present one or more null columns and therefore fail to yield tripartite factorizations. Looking to the remaining four, we have

$$\begin{aligned}\mathbb{M}_1&=\begin{pmatrix}3&1&1\\2&0&0\end{pmatrix}\Longrightarrow p_1^3p_2^2\cdot p_1^1\cdot p_1^1\\\mathbb{M}_2&=\begin{pmatrix}2&2&1\\2&0&0\end{pmatrix}\Longrightarrow p_1^2p_2^2\cdot p_1^2\cdot p_1^1\\\mathbb{M}_3&=\begin{pmatrix}3&1&1\\1&1&0\end{pmatrix}\Longrightarrow p_1^3p_2^1\cdot p_1^1p_2^1\cdot p_1^1\\\mathbb{M}_4&=\begin{pmatrix}2&2&1\\1&1&0\end{pmatrix}\Longrightarrow p_1^2p_2^1\cdot p_1^2p_2^1\cdot p_1^1\end{aligned}$$

This list of tripartite factorizations is, however, not complete, for the rows have been presented in the descending lexicographic order in which they were produced by *Mathematica*, and this is a convention extraneous to the problem at hand. Look, for example to

$$\mathbb{M}_5=\begin{pmatrix}1&2&2\\2&0&0\end{pmatrix}\Longrightarrow p_1^1p_2^2\cdot p_1^2\cdot p_1^2$$

which obtains from $\mathbb{M}_2$ by permutation of the top row. And a permutation of the top row of a matrix that we initially were led to abandon sends

$$\begin{pmatrix}4&1&0\\1&1&0\end{pmatrix}\longrightarrow\begin{pmatrix}4&0&1\\1&1&0\end{pmatrix}\Longrightarrow p_1^4p_2^1\cdot p_2^1\cdot p_1^1$$

which is a tripartite factorization absent from the list accumulated thus far. It appears we must expand the $\mathbb{M}$-set (7) by subjecting each row to all distinct permutations, then discard all matrices with one or more null columns, then discard all duplicates. $F(n,m)$ is the number of matrices that survive that process. $f(n,m)$ results when one filters out matrices with permutationally equivalent columns. *Mathematica* can be used to implement the winnowing processes described above, but I have found that to be a challenging exercise.

Concluding remarks. In 1983 (the year that the Canfield/Erdös/Pomerance paper[7] appeared), J. F. Hughes & J. O. Shallit published a short paper entitled "On the number of multiplicative partitions."[9] The Wikipedia "multiplicative partitions" article states that Hughes & Shallit appear to be responsible for the fact that the "factorissatio numerorum problem" has come to be called the "multiplicative partitions problem" (sometimes the "unordered factorization problem.") We see that the new terminology could not be more apt, though it seems a shame to abandon one of the last surviving Latinisms in mathematical parlance.

When I first approached this problem I anticipated that it might yield to a generating function technique, but came away empty handed (where are Euler and Ramanujan when we need them?). Recently I came across a paper by Shamik Ghosh[10] the abstract of which reads

> *...we describe a new method of counting the number of unordered factorizations of a natural number by means of a generating function and a recurrence arising from it, which improves an earlier result in this direction.*

But Ghosh is—like most authors in this field—concerned with *counting*, while I seek *explicit lists* of m-partite factorizations.

ADDENDA: Two exceptionally simple cases. Suppose n is a power of a prime:

$$n = p^\alpha$$

The multiplicative partitions problem reduces then to the problem of literally partitioning α. Suppose, for example, that $\alpha = 6$. Then

$$\begin{aligned}
\texttt{IntegerPartitions[6,\{2\}]} &= \Big\{\{5,1\},\{4,2\},\{3,3\}\Big\}\\
\texttt{IntegerPartitions[6,\{3\}]} &= \Big\{\{4,1,1\},\{3,2,1\},\{2,2,2\}\Big\}\\
\texttt{IntegerPartitions[6,\{4\}]} &= \Big\{\{3,1,1,1\},\{2,2,1,1\}\Big\}\\
\texttt{IntegerPartitions[6,\{5\}]} &= \quad \{2,1,1,1,1\}\\
\texttt{IntegerPartitions[6,\{6\}]} &= \quad \{1,1,1,1,1,1\}
\end{aligned}$$

from which unordered m-partite factorizations can be read off directly and ordered factorizations obtained permutationally.

In quantum mechanical applications one is—as was remarked previously—most commonly concerned with bipartite factorizations

$$n = q_1 q_2 = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_\nu^{\alpha_\nu}$$

[9] American Mathematical Monthly **90**, 468-471.

[10] "Counting number of factorizations of a natural number," arXiv:0811.3479v1 (21 November 2008).

The simplifying feature here is that the structure of $q_2 = n/q_1$ is implicit in that of q_1. The possible bipartite factorizations are

$$\begin{aligned} q_1 &= p_1^{\mu_1} p_2^{\mu_2} \cdots p_\nu^{\mu_\nu} \\ q_2 &= p_1^{\alpha_1-\mu_1} p_2^{\alpha_2-\mu_2} \cdots p_\nu^{\alpha_\nu-\mu_\nu} \end{aligned}$$

where

$$\begin{aligned} \mu_1 &\in \{0,1,2,\ldots,\alpha_1\} \\ \mu_2 &\in \{0,1,2,\ldots,\alpha_2\} \\ &\vdots \\ \mu_\nu &\in \{0,1,2,\ldots,\alpha_\nu\} \end{aligned}$$

There are a total of $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_\nu+1)$ such μ-assignments, of which two (all μ minimal and all μ maximal) must be excluded because they produce $q_1 = 1$ else $q_2 = 1$ and so violate the bipartite requirement. So we have

$$\begin{aligned} F(n,2) &= (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_\nu+1) - 2 \\ f(n,2) &= \texttt{Ceiling[}\tfrac{1}{2}F(n,2)\texttt{]} \end{aligned}$$

EXAMPLE: In cases of the form $n = p_1^2 p_2^1$ we have

$$q_1 = [p_1^0 p_2^0],\ p_1^0 p_2^1,\ p_1^1 p_2^0,\ p_1^1 p_2^1,\ p_1^2 p_2^0,\ [p_1^2 p_2^1]$$

where the bracketed cases must be omitted. If $n = 12 = 2^2 \cdot 3$ (the smallest instance of such a case) we have the bipartite factorization list

$$12 = 3\cdot 4 = 2\cdot 6 = 6\cdot 2 = 4\cdot 3$$

which are

$$F(12,2) = (2+1)(1+1) - 2 = 4$$

in number. If order doesn't count we have

$$12 = 3\cdot 4 = 2\cdot 6$$

which are

$$f(12,2) = \texttt{Ceiling[}\tfrac{1}{2}F(12,2)\texttt{]} = 2$$

in number.

EXAMPLE: In cases of the form $n = p_1^2 p_2^2$ we have

$$q_1 = [p_1^0 p_2^0],\ p_1^0 p_2^1,\ p_1^0 p_2^2,\ p_1^1 p_2^0,\ p_1^1 p_2^1,\ p_1^1 p_2^2,\ p_1^2 p_2^0,\ p_1^2 p_2^1,\ [p_1^2 p_2^2]$$

which in the leading case $36 = 2^2 \cdot 3^2$ gives

$$36 = 3\cdot 12 = 9\cdot 4 = 2\cdot 18 = 6\cdot 6 = 18\cdot 2 = 4\cdot 9 = 12\cdot 3$$

$$F(36,2) = (2+1)(2+1) - 2 = 7$$

while if order doesn't count we have

$$36 = 3 \cdot 12 = 9 \cdot 4 = 2 \cdot 18 = 6 \cdot 6$$

$$f(36,2) = \texttt{Ceiling[}\tfrac{1}{2} \cdot 7\texttt{]} = 4$$

The *Mathematica* commands

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F[n_]:=Product[FactorInteger[n][[k]][[2]]+1,{k,1,Length[FactorInteger[n]]}]−2
f[n_]:=Ceiling[F[n]/2]
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(which announce 0 when n is prime) permit one to tabulate results such as those we just computed by hand. We have, for example, this short table of bipartite factorization numbers:

n	$F(n,m)$	$f(n,m)$
4838398	6	3
4838399	0	0
4838400	262	131
4838401	2	1
4838402	6	3

where the number $4838400 = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7$ was taken from the "Table of highly factorable integers below 10^9" that appears in Canfield *et al.*[7] Similarly

n	$F(n,m)$	$f(n,m)$
958003198	2	1
958003199	14	7
958003200	862	431
958003201	0	0
958003202	10	5

where $958003200 = 2^{11} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$ is the last entry in the Canfield table.

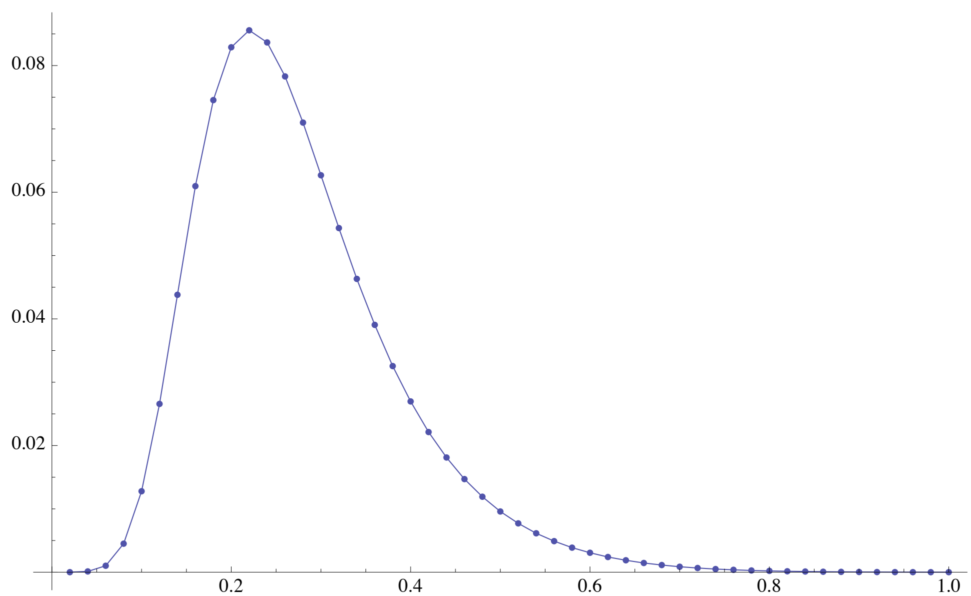
As we have seen, the theory of multiplicative partitions hinges on a subject with a much longer history—the theory of additive partitions, as it relates to the prime exponents. Which recalls to mind a problem that engaged the attention of Richard Crandall and me for a few days exactly five years ago (February, 2008). Let

$$p(n,m) = \#\ m\text{-term additive partitions of } n$$

and define

$$P(n,m) = \frac{p(n,m)}{p(n)} \quad : \quad \sum_{m=1}^{n} P(n,m) = 1$$

We were intrigued by the observation that when plotted (see the following figure) $P(n,m)$ resembles some distributions (Maxwell, Planck) of physical importance, and sought asymptotic analytical approximations to $P(n,m)$. It was brought to Richard's attention (by Carl Pomerance?) that the problem has generated a literature : see P. Erdös & J. Lehmer, "The distribution of the number of summands in the partitions of a positive integer," Duke Math. J. **8**, 335-345 (1941); G. Szekeres, "Some asymptotic formulae in the theory of partitions (II)," Quarterly J. of Math. (Oxford) **4**, 96-111 (1953).



$$\frac{\texttt{Length[IntegerPartitions[50, \{m\}]]}}{\texttt{PartitionsP[50]}}$$

Memorandum. The Crandall connection led me to send a copy of this essay to Carl Pomerance,[7] who promptly suggested that I might "try writing to Paul Pollack who has worked on multiplicative partitions fair recently, or Florian Luca," and supplied their email addresses. I did so, and within a few days received (28 Februry = yesterday) a note from Pollack reporting that he was "not an expert in the computational aspects of these problems...the people in the know are Arnold Knopfmacher and Michael Mays; the most relevant recent paper" is http://www.math.wvu.edu/∼mays/Papers/Factorizations.pdf (Mays is at the University of West Virginia). The paper in question is "Ordered and unordered factorizations of integers," The Mathematica Journal **10**, 72-89 (2006). The paper in question—repleat with detailed *Mathematica* commands—very nicely addresses all aspects of the problem explored in this naive essay. I have included a copy in this file.